1 Introduction

Modern vector graphic programs like Inkscape make heavy use of Bézier curves to display graphs. We attempt to replace a curve consisting of piecewise Bézier curves by a single Bézier curve in order to simplify the graph while maintaining the shape as good as possible. This is useful for simpifying unneccessary complex graphs created by automatic vectorization of bitmaps. Therefore we formulate an optimization problem: Given a Funktion F given by a piecewise Bézier curve $(S^1, S^2, \cdots, S^n)$, each Bézier curve described by d control points $(P_1^1, P_2^1, \cdots, P_d^1, P_1^2, P_2^2, \cdots, \cdots, P_d^n)$ find the Bézier curve G described by control points $(G_1, \cdots, G_s)$ which minimizes the following target function:

$$\Phi(G_1, \cdots, G_s) = \int_0^1 \|F(t) - G(t)\|_2^2$$

This integral will be evaluated by numerical integration despite the fact that the integrand is a polynomial because it is difficult to parameterise Bézier curves by arc length.

We are interested in using gradient based methods for faster optimization, therefore we calculate derivatives:

$$\frac{\partial}{\partial G_k} \Phi(G_1, \cdots, G_k) = \frac{\partial}{\partial G_k} \int_0^1 \|F(t) - G(t)\|_2^2$$

F, G and $\|\cdot\|_2^2$ are continous and all partial derivatives are continous, therefore we may interchange derivative and integral:

$$\frac{\partial}{\partial G_k} \Phi(G_1, \cdots, G_s) = \int_0^1 \frac{\partial}{\partial G_k} \|F(t) - G(t)\|_2^2$$

We need some lemmata:

$$\begin{aligned}
\frac{\partial}{\partial x}\|x\|_2^2 &= \frac{\partial}{\partial x}\sum_{k=1}^{n} x_k^2 = 2x \\
B_{l,s}(t) &:= \binom{s}{l} t^l (1-t)^{s-l} \\
\frac{\partial}{\partial G_k}[F(t) - G(t)] &= \frac{\partial}{\partial G_k}\left[F(t) - \sum_{l=0}^{s} B_{l,s}(t) G_l\right] \\
&= 0 - \frac{\partial}{\partial G_k}\sum_{l=0}^{s} B_{l,s}(t) G_l \\
&= -\sum_{l=0}^{s} \frac{\partial}{\partial G_k} B_{l,s}(t) G_l \\
&= -\frac{\partial}{\partial G_k} B_{k,s}(t) G_k \\
&= -B_{k,s}(t)
\end{aligned}$$

Now we put it all together to calculate the derivative we are interested in:

$$\begin{aligned}
\int_0^1 \frac{\partial}{\partial G_k}\|F(t) - G(t)\|_2^2 &= \int_0^1 2(F(t) - G(t)) \cdot (-B_{k,s}(t)) \\
&= -2\int_0^1 (F(t) - G(t)) \cdot B_{k,s}(t)
\end{aligned}$$

This can be easily computed while integrating $\|F(t) - G(t)\|_2^2$.